Advanced Placement Calculus Name _____

Copy original problem. Per _____ Date _____

Convince *me* that **you** understand the concept!

Final Exam

I Given $f(x)=\begin{cases} 8\sqrt{x-1} & 1\le x\le 5 \\ (x-a)^2+b & x>5 \end{cases}$.

What values must be assigned to a and b in order that $f(x)$ be differentiable for all $x>1$. (15 pts)

II The region bounded by the parabolas $y=x^2$ and $y = 6x - x^2$ is rotated about the x–axis so that a vertical line segment cut off by the curves generates a ring. Find the value of x for which the ring of largest area is obtained. (15 pts)

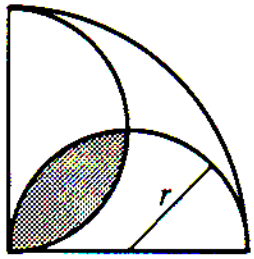
III Let $f(x)=12-x^2$ for $x\ge 0$ and $f(x)\ge 0$. (10 pts ea)

a) The line tangent to the graph of $f(x)$ at the point $(k, f(k))$ intercepts the x–axis at $x=4$. What is the value of k?

b) An isosceles triangle whose base is the interval from $(0,0)$ to $(c,0)$ has its vertex on the graph of $f(x)$. For what value of c does the triangle have maximum area? Justify, of course.

IV Inside this quarter circle are two semi-circles having the same radius r. Let the area common to all 3 circle portions be R. Compute the area of R.

Use: $\int \sqrt{2ax-x^2}\,dx=\frac{1}{2}\left[(x-a)+a^2\sin^{-1}\left(\frac{x-a}{|a|}\right)\right]$



(25 pts)

V Let f and g be continuous functions with the following properties:

i $g(x)=A-f(x)$

ii $\int_1^2 f(x)\,dx=\int_2^3 g(x)\,dx$

iii $\int_2^3 f(x)\,dx=-3A$

(25 pts tot)

a) Determine $\int_1^3 f(x)\,dx$ in terms of A.

b) Find the average value, in terms of A, of $g(x)$ over [1 , 3].

c) Given $\int_0^1 f(x+1)\,dx=kA$. Solve for k.

Extra Credit ---------- 5 pts ----------

A curve is given parametrically by the equations $x=3-2\sin t$ and $y=2\cos t-1$. Find the length of the arc from $t=0$ to $t=\pi$.

I $f(x) = \begin{cases} 8\sqrt{x-1} & 1 \le x \le 5 \\ (x-a)^2 + b & x > 1 \end{cases}$

To be differentiable, it must be continuous.

$f(5) = 8\sqrt{5-1} = 16$

$\lim_{x \to 5^-} 8\sqrt{x-1} = 16$

$\lim_{x \to 5^+} (x-a)^2 + b = (5-a)^2 + b$

$(5-a)^2 + b \overset{\text{set}}{=} 16$

$b = 16 - (5-a)^2$

$10 - 2a \overset{\text{set}}{=} 2$

$-2a = -8$

$a = 4$

$b = 16 - 1^2$

$b = 15$

a is 4, b is 15.

$\lim_{x \to 5^-} \dfrac{8\sqrt{x-1} - 16}{x-5}$

$\lim_{x \to 5^-} \dfrac{8(\sqrt{x-1} - 2)}{x-5} \dfrac{\sqrt{\ } + 2}{\sqrt{\ } + 2}$

$\lim_{x \to 5^-} \dfrac{8(x-5)}{(x-5)(\sqrt{x-1} + 2)} = \dfrac{8}{4} = 2$

$\lim_{x \to 5^+} \dfrac{(x-a)^2 + b - 16}{x-5}$

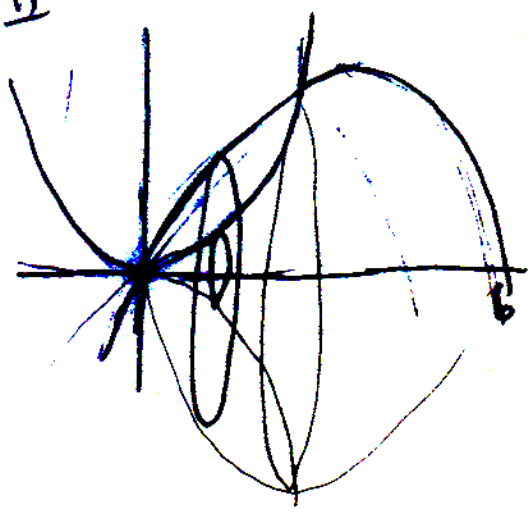
$\lim_{x \to 5^+} \dfrac{(x-a)^2 + 16 - (5-a)^2 - 16}{x-5}$

$\lim_{x \to 5^+} \dfrac{(x-a)^2 - (5-a)^2}{x-5}$

$\lim_{x \to 5^+} \dfrac{((x-a) - (5-a))((x-a) + (5-a))}{x-5}$

$\lim_{x \to 5^+} \dfrac{(x-5)(x+5-2a)}{x-5} = 10 - 2a$

II



$y = x^2$

$y = 6x - x^2$

$A(x) = \pi\left[(6x - x^2)^2 - (x^2)^2\right]$

$= \pi\left[36x^2 - 12x^3 + x^4 - x^4\right]$

$= \pi\left[36x^2 - 12x^3\right]$

$A(x) = 12x^2\pi[3 - x]$

$A' = \pi[72x - 36x^2]$

$36x\pi[2 - x] \overset{\text{set}}{=} 0$

$x = 2$

$A'' = \pi(72 - 72x)$

$72\pi(1-x) < 0$ for $x > 2$

$\therefore$ max f'' test.

The rectangle with the greatest area occurs when $x = 2$.

III $f(x) = 12 - x^2$

a) $f'(x) = -2x$

$(k, 12-k^2)$

$(4, 0)$

$\frac{12-k^2}{k-4} \overset{\text{set}}{=} -2k$

$12 - k^2 = -2k^2 - 8k$

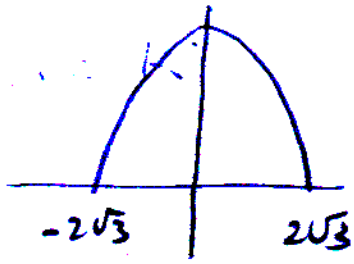
$k^2 - 8k + 12 = 0$

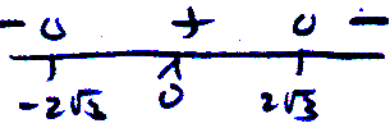
$(k-2)(k-6) = 0$

$k \in \{2, 6\}$

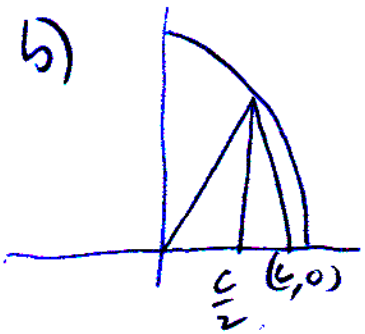
Reject $k=6$ since $f \geq 0$

So $k = 2$.





b)



$A = \frac{1}{2} bh \quad A(c) = \frac{1}{2} c \left(12 - \frac{c^2}{4}\right)$

$= 6c - \frac{c^3}{8}$

$A' = 6 - \frac{3c^2}{8} \overset{\text{set}}{=} 0$

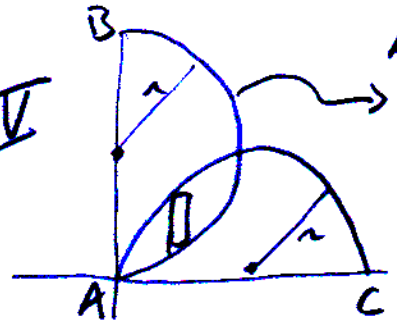
$3c^2 = 48$

$c = 4$

$A'' = -\frac{3c}{4} < 0$

max f'' test.

IV



ARC $\overset{\frown}{AB}$

$x^2 + (y-r)^2 = r^2$

$y - r = \pm\sqrt{r^2 - x^2}$

$y = r - \sqrt{r^2 - x^2}$

↑ Notice we need bottom of this circle

ARC $\overset{\frown}{AC}$ $(x-r)^2 + y^2 = r^2$

$y = \sqrt{r^2 - (x-r)^2}$

$= \sqrt{r^2 - x^2 + 2rx - r^2}$

$y = \sqrt{2rx - x^2}$

intersection point?

$\overset{\frown}{AC}$

$x^2 - 2rx + r^2 + y^2 = r^2$

$x^2 + y^2 = 2rx$

$\overset{\frown}{AB}$ $x^2 + y^2 - 2ry + r^2 = r^2$

$x^2 + y^2 = 2ry$

$2rx = 2ry \therefore x = y$

$x^2 = rx$

$x(x-r) = 0$

$x = r$

$\int_0^r \sqrt{2rx - x^2} - (r - \sqrt{r^2 - x^2})\,dx$

$\int_0^r \sqrt{2rx - x^2}\,dx - \int_0^r r\,dx + \int_0^r \sqrt{r^2 - x^2}\,dx$

① ② ③

using hint

① $\int_0^r \sqrt{2rx - x^2}\,dx = \frac{1}{2}\left[(x-r)\sqrt{2rx - x^2} + r^2 \sin^{-1}\left(\frac{x-r}{|r|}\right)\right]_0^r$

$= \frac{1}{2}\left[0 + r^2 \cdot 0 - \left[-r \cdot 0 + r^2 \sin^{-1}(-1)\right]\right]$

$= \frac{1}{2}\left[\frac{\pi r^2}{2}\right] = \frac{\pi r^2}{4}$

② $\int_0^r r\,dx = rx\Big|_0^r = r^2$

continued ↓

IV ③ $\int_0^1 \sqrt{1-x^2}\,dx$

$\sin\theta = \frac{x}{1}$ $\quad x=0 \quad \theta=0$

$1\sin\theta = x$ $\quad x=1 \quad \theta=\frac{\pi}{2}$

$1\cos\theta\,d\theta = dx$

$$\int_0^{\frac{\pi}{2}} 1^2\cos^2\theta\,d\theta = \frac{1}{2}1^2\int_0^{\frac{\pi}{2}}(1+\cos 2\theta)\,d\theta$$

$$= \frac{1}{2}1^2\left[\theta + \frac{1}{2}\sin 2\theta\right]\Big|_0^{\frac{\pi}{2}} = \frac{1^2}{2}\left[\frac{\pi}{2} + \frac{1}{2}\cdot 0 - 0\right]$$

$$= \frac{\pi 1^2}{4}$$

all ~~together~~ ① ② ③

$$\frac{\pi 1^2}{4} - 1^2 + \frac{\pi 1^2}{4} = \frac{\pi 1^2}{2} - 1^2 = \frac{1^2}{2}(\pi - 2) = \text{area}$$

V a) $\int_1^3 f(x)\,dx = \underbrace{\int_1^2 f(x)\,dx}_{ii} + \int_2^3 f(x)\,dx$ (i)

$$\int_2^3 g(x)\,dx + \int_2^3 (A - g(x))\,dx$$

$$+ \int_2^3 A\,dx - \int_2^3 g(x)\,dx = \int_2^3 A\,dx = Ax\Big|_2^3 = A.$$

b) ave value

$$\frac{1}{3-1}\int_1^3 g(x)\,dx$$

$$\frac{1}{2}\int_1^3 g(x)\,dx = \frac{1}{2}\left[\int_1^2 g(x)\,dx + \int_2^3 g(x)\,dx\right] = \frac{1}{2}\left[\int_1^2 A - f(x)\,dx + \int_2^3 g(x)\,dx\right] =$$

$$= \int_1^2 A\,dx - \int_1^2 f(x)\,dx + \int_2^3 g(x)\,dx \qquad = \frac{1}{2}\int_1^2 A\,dx = \frac{Ax}{2}\Big|_1^2 = \frac{A}{2}$$

$\int_2^3 g(x)\,dx$

c) $\int_0^1 f(x+1)\,dx = KA$

$$\int_1^2 f(u)\,du = \int_2^3 g(u)\,du = \int_2^3 A - f(u)\,du =$$

let $x+1=u$ $\quad x=0 \; u=1$

$dx = du$ $\quad x=1 \; u=2$

$$\int_2^3 A\,du - \int_2^3 f(u)\,du = Au\Big|_2^3 - (-3A) =$$

$$\int_1^2 f(u)\,du = KA \qquad A + 3A = 4A \overset{\text{set}}{=} KA$$

$\therefore K = 4$

EXTRA $\int_0^{\pi} \sqrt{\dot{x}^2 + \dot{y}^2}\,dt$ $\quad x = 3 - 2\sin t$ $\quad y = 2\cos t - 1$ $\quad \sqrt{\dot{x}^2+\dot{y}^2} = \sqrt{4} = 2$

$\dot{x} = -2\cos t$ $\quad \dot{y} = -2\sin t$

$\dot{x}^2 = 4\cos^2 t$ $\quad \dot{y}^2 = 4\sin^2 t$ $\quad \int_0^{\pi} 2\,dt = 2t\Big|_0^{\pi} = 2\pi.$